Math 2413
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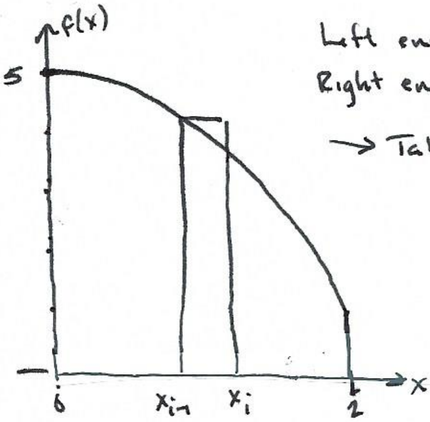
Ex. Find the area of the region by definition.

2) $f(x) = -x^2+5$ on $[0,2]$. Take sample points to be left endpoints. Sketch the region.



Left endpoints are $c_i = 0 + (i-1)\Delta x = (i-1)\Delta x$

Right endpoints are $c_i = 0 + i\Delta x = i\Delta x$

* $\Delta x = \frac{b-a}{n} = \frac{2}{n}$

→ Taking left endpoints: $x_0 = 0,\ x_1 = \Delta x,\ x_2 = 2\Delta x, \ldots, x_n = n\Delta x = 2$

$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f[(i-1)\Delta x]\Delta x = \lim_{n\to\infty} -\frac{2}{n}\sum_{i=1}^{n}\left[(i-1)^2\frac{4}{n^2} - 5\right]$$

$$= \lim_{n\to\infty} -\frac{2}{n}\sum_{i=1}^{n}\left[\frac{4}{n^2}(i^2-2i+1) - 5\right]$$

$$= \lim_{n\to\infty} -\frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 + \frac{4}{n^2}\sum_{i=1}^{n} 2i - \frac{4}{n^2}\sum_{i=1}^{n}(1) - \sum_{i=1}^{n}(5)\right]$$

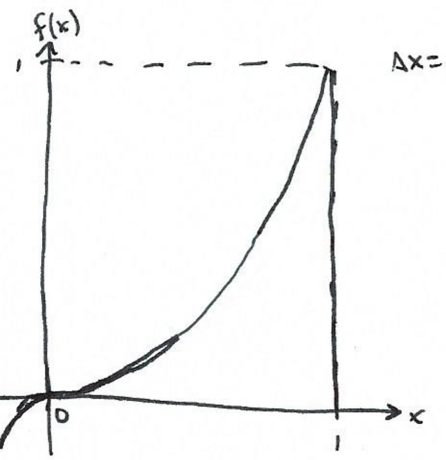
$$= \lim_{n\to\infty} -\frac{2}{n}\left[\frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{4}{n^2}\cdot 2\cdot\frac{n(n+1)}{2} - \frac{4}{n^2}n - 5n\right]$$

$$= \lim_{n\to\infty}\left[\frac{-8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{8}{n^2}(n+1) - \frac{8}{n^2} + 10\right]$$

$$= \lim_{n\to\infty}\left[-\frac{4(n+1)(2n+1)}{3n^2} + \frac{8(n+1)}{n^2} - \frac{8}{n^2} + 10\right]$$

$$= -\frac{8}{3} + 10 = \frac{22}{3}$$

3) $f(x) = x^3$ on $[0,1]$. Sketch the region.

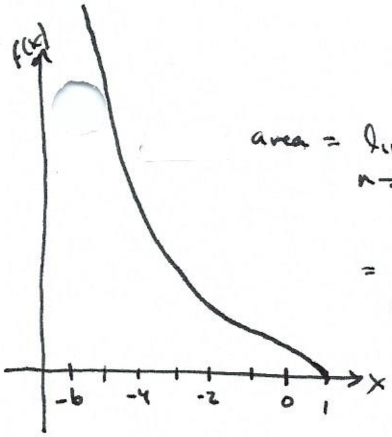


$\Delta x = \frac{1-0}{n} = \frac{1}{n}$ right endpoints: $0 + i\Delta x = \frac{i}{n} = c_i$

$$\text{area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \lim_{x\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i^3}{n^3} = \lim_{x\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \lim_{n\to\infty}\frac{(n+1)^2}{4n^2} = \frac{1}{4}$$

4) $f(x) = -x^3 + 1$ on $[-6,-4]$



$$\Delta x = \frac{-4 - -6}{n} = \frac{2}{n}$$

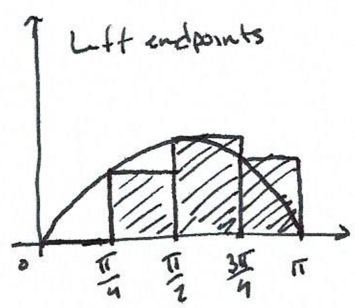
right endpoints: $c_i = -6 + i\Delta x = -6 + \frac{2i}{n}$

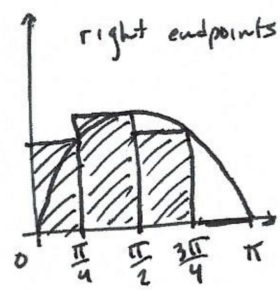
$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left[-\left(-6+\frac{2i}{n}\right)^3 + 1\right]$$

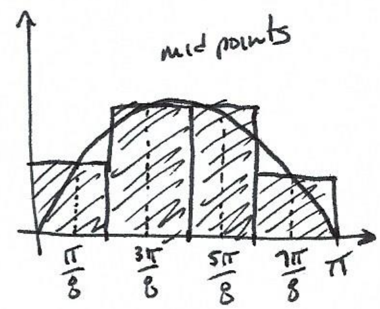
$$= \lim_{n\to\infty} \frac{2}{n}\left[-\sum_{i=1}^{n}\left(-6+\frac{2i}{n}\right)^3 + \sum_{i=1}^{n}(1)\right] = \lim_{n\to\infty}\frac{2}{n}\left[-\sum_{i=1}^{n}\left((-6)^3 + 3(-6)^2\left(\frac{2i}{n}\right) + 3(-6)\left(\frac{2i}{n}\right)^2 + \left(\frac{2i}{n}\right)^3\right) + n\right]$$

Thm Midpoint rule Let $c_i = \frac{x_i + x_{i-1}}{2}$. Then $\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$

Ex Use the midpoint rule with $n=4$ to approximate the area of the region bounded by the graph of $f(x) = \sin(x)$ and the x-axis for $0 \le x \le \pi$.







$$\text{Area} \approx \sum_{i=1}^{4} f(c_i)\Delta x = f\left(\frac{\pi}{8}\right)\cdot\frac{\pi}{4} + f\left(\frac{3\pi}{8}\right)\cdot\frac{\pi}{4} + f\left(\frac{5\pi}{8}\right)\cdot\frac{\pi}{4} + f\left(\frac{7\pi}{8}\right)\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{4}\left(\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right)\right)$$