Math 2413
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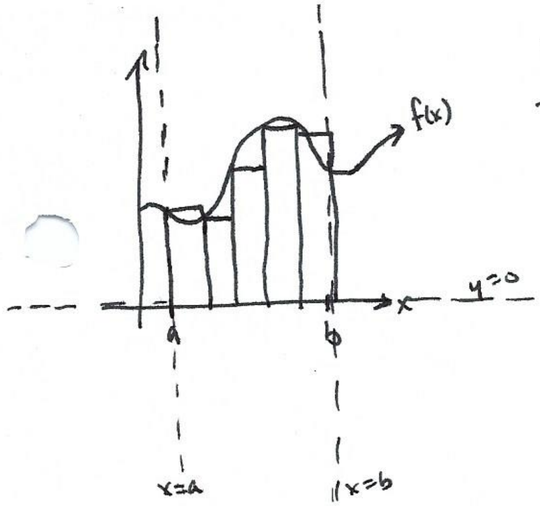
Pg 68



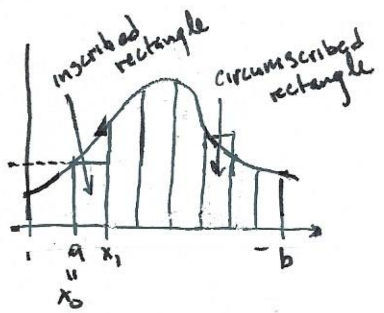
Let f be non-negative, continuous function on $[a,b]$.

To approximate the area of the region bounded by $f(x)$, the x-axis, and the vertical lines $x=a$ and $x=b$:

divide the interval $[a,b]$ into n subintervals with length $\Delta x = \frac{b-a}{n}$ and endpoints $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x, \ldots, x_n = a + n\Delta x = b$

Since f continuous, the Extreme Value Theorem guarantees a max and min of f on each subinterval.

Let $f(m_i)$ and $f(M_i)$ be the min and max values of f on the i^{th} subinterval.

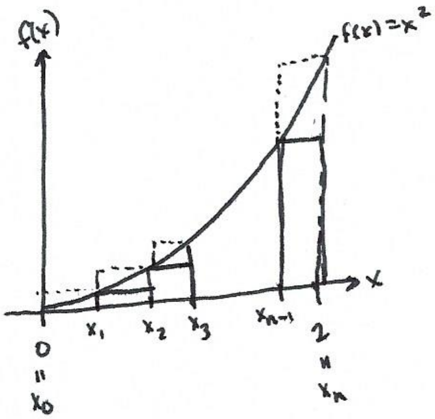


Area of ith rectangle: $f(m_i)\Delta x \le f(M_i)\Delta x$

Lower sum $= \sum_{i=1}^{n} f(m_i)\Delta x$ ← area of inscribed rectangles

Upper Sum $= \sum_{i=1}^{n} f(M_i)\Delta x$ ← area of circumscribed rectangles

Ex Find upper and lower sums for region bounded by $f(x) = x^2$ and the x-axis between $x=0$ and $x=2$.



$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$, $x_0 = a$, $x_1 = a + \Delta x, \ldots x_n = a + n\Delta x$

Since f is increasing on $[0,2]$
$m_i = x_{i-1} = a + (i-1)\frac{2}{n}$ ← left end points
$M_i = x_i = a + i\frac{2}{n}$ ← right endpoints

Lower sum: $= S(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} f\left(a + (i-1)\frac{2}{n}\right)\frac{2}{n}$, $(a=0)$

$= \frac{2}{n}\sum_{i=1}^{n} f\left((i-1)\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left[(i-1)\frac{2}{n}\right]^2 = \frac{2}{n}\sum_{i=1}^{n}(i-1)^2\frac{4}{n^2}$

$= \frac{8}{n^3}\sum_{i=1}^{n}(i^2 - 2i + 1) = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right]$

$= \frac{8}{n^2}\left[\frac{(n+1)(2n+1)}{6} - (n+1) + 1\right] = \frac{8}{n^2}\left[\frac{(n+1)(2n+1) - 6n}{6}\right]$

$= \frac{8}{n^2}\left[\frac{2n^2 + n + 2n + 1 - 6n}{6}\right] = \frac{8}{6n^2}\left[2n^2 - 3n + 1\right] = \ldots = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$

Ex. (cont.)

Upper sum: $\square = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n}$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2 = \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{4}{3n^2}\left(2n^2+3n+1\right)$$

$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

if The actual is A, then $s(n) \le A \le S(n)$

$$\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} \le A \le \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

$\lim_{n\to\infty} s(n) = \frac{8}{3} = \lim_{n\to\infty} S(n)$, so the actual area $A = \frac{8}{3}$.

Thm. If f is continuous and non-negative on $[a,b]$, then

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n),$$

where $\Delta x = \frac{b-a}{n}$ and m_i and M_i are the max and min values of on ith subinterval.

Def. (Area in the plane): Let f be continuous, non-negative on $[a,b]$. The area of the region bounded by $\begin{cases} x=a \\ x=b \\ y=0 \\ f(x) \end{cases}$ is $A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$, where $x_{i-1} \le c_i \le x_i$ and $\Delta x = \frac{b-a}{n}$.

c_i is called a sample point.

Ex. Find the area of the region by definition.

1) $f(x) = x^2$ from $x=0$ to $x=2$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$$