Let $f$ be a non-negative, continuous function on $[a, b]$. To approximate the area of the region bounded by $f(x)$, the $x$-axis, and the vertical lines $x = a$ and $x = b$, divide the interval $[a, b]$ into $n$ subintervals with length $\Delta x = \frac{b-a}{n}$ and endpoints $x_0 = a, \ x_1 = a + \Delta x, \ldots, x_n = a + n\Delta x$.

Since $f$ is continuous, the Extreme Value Theorem guarantees a max and min of $f$ on each subinterval. Let $f(m_i)$ and $f(M_i)$ be the min and max values of $f$ on the $i$th subinterval.

Area of $i$th rectangle: $f(m_i)\Delta x \leq f(M_i)\Delta x$

Lower sum: $\sum_{i=1}^{n} f(m_i)\Delta x = \text{area of inscribed rectangles}$

Upper sum: $\sum_{i=1}^{n} f(M_i)\Delta x = \text{area of circumscribed rectangles}$

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Ex. Find upper and lower sums for region bounded by $f(x) = x^2$ and the $x$-axis between $x = 0$ and $x = 2$.

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$, $x_0 = a, \ x_1 = a + \Delta x, \ldots, x_n = a + n\Delta x$

Since $f$ is increasing, $M_i = x_i-1 = a + (i-1)\frac{2}{n}$ \leftarrow \text{right endpoints on } [0, 2]$

$L_i = x_i = a + i\frac{2}{n}$ \leftarrow \text{left endpoints}$

Lower sum: $L = \sum_{i=1}^{n} f(L_i)\Delta x = \sum_{i=1}^{n} f(a + i\frac{2}{n})\frac{2}{n}$, $a = 0$

$= \frac{2}{n} \left[ \sum_{i=1}^{n} f(i\frac{2}{n}) \right] = \frac{2}{n} \sum_{i=1}^{n} \left( i\frac{2}{n} \right)^2 = \frac{2}{n} \sum_{i=1}^{n} \frac{i^2 \cdot 4}{n^2}$

$= \frac{8}{n^3} \sum_{i=1}^{n} (i^2 - 1) = \frac{8}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} - (n+1) \right]$.

$= \frac{8}{n^3} \left[ \frac{2n^3 + 3n^2 + n}{6} \right] = \frac{8}{3n} \left[ \frac{2n^2 - 3n + 1}{3n} \right] = \frac{8}{3n} - \frac{4}{n} + \frac{4}{3n}$.
Ex. (cont.)

Upper sum: \( S(n) = \sum_{i=1}^{n} f(M_i) \Delta x = \sum_{i=1}^{n} f \left( \frac{2i}{n} \right) \frac{2}{n} \)

\[ = \frac{2}{n} \sum_{i=1}^{n} \left( \frac{2i}{n} \right)^2 = \frac{2}{n^3} \sum_{i=1}^{n} i^2 = \frac{2}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right) = \frac{4}{3n^2} \left( \frac{2n^2 + 3n + 1}{n^2} \right) \]

\[ = \frac{8}{3} + \frac{4}{n} + \frac{3}{3n^2} \]

If the actual is \( A \), then \( S(n) \leq A \leq S(n) \)

\[ \frac{8}{3} + \frac{4}{n} + \frac{3}{3n^2} \leq A \leq \frac{8}{3} + \frac{4}{n} + \frac{3}{3n^2} \]

\[ \lim_{n \to \infty} S(n) = \frac{8}{3} = \lim_{n \to \infty} S(n) \]

so the actual area \( A = \frac{8}{3} \).

Thus. If \( f \) is continuous and non-negative on \([a,b]\), then

\[ \lim_{n \to \infty} S(n) = \lim_{n \to \infty} \sum_{i=1}^{n} f(M_i) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(M_i) \Delta x = \lim_{n \to \infty} S(n) \]

where \( \Delta x = \frac{b-a}{n} \) and \( M_i \) and \( m_i \) are the max and min values of \( f \) on \( i \)th subinterval.

Def. (Area in the plane): Let \( f \) be continuous, non-negative on \([a,b]\). The area of the region bounded by

\[ \int_{x=a}^{x=b} \begin{cases} 0 & \text{if } y \geq f(x) \ \text{and } x \leq a \ \text{or } x \geq b \\\n \end{cases} \]

is \( A = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x \), where \( x_{i-1} \leq c_i \leq x_i \) and \( \Delta x = \frac{b-a}{n} \).

\( c_i \) is called a sample point.

Ex. Find the area of the region by definition.

1) \( f(x) = x^2 \) from \( x=0 \) to \( x=2 \)

\[ \text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x = \lim_{n \to \infty} \left( \frac{8}{3} - \frac{4}{n} + \frac{3}{3n^2} \right) = \frac{8}{3} \]