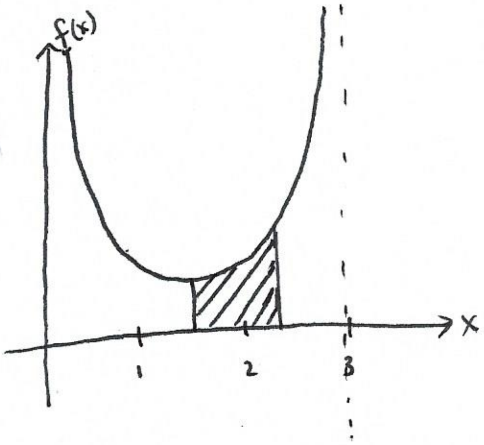
Ex. Find the area of the region bounded by $f(x) = \dfrac{1}{\sqrt{3x - x^2}}$, the x-axis, and $\begin{cases} x = \frac{3}{2} \\ x = \frac{9}{4} \end{cases}$




$$\int_{3/2}^{9/4} \frac{dx}{\sqrt{3x - x^2}} = \int_{3/2}^{9/4} \frac{dx}{\sqrt{-(x^2 - 3x)}} = \int_{3/2}^{9/4} \frac{dx}{\sqrt{-\left[x^2 - 3x + \left(\frac{3}{2}\right)^2\right] + \left(\frac{3}{2}\right)^2}}$$

$$= \int_{3/2}^{9/4} \frac{dx}{\sqrt{-\left(x - \frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2}},$$

let $u = x - \frac{3}{2}$, $du = dx$

$x = \frac{3}{2} \rightarrow u = 0$

$x = 9/4 \rightarrow u = \frac{3}{4}$

$$= \int_0^{3/4} \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - u^2}} = \arcsin\left(\frac{u}{3/2}\right)\Big|_0^{3/4} = \arcsin\left(\frac{3/4}{3/2}\right) - \arcsin(0)$$

$$= \arcsin\left(\tfrac{1}{2}\right) - 0$$

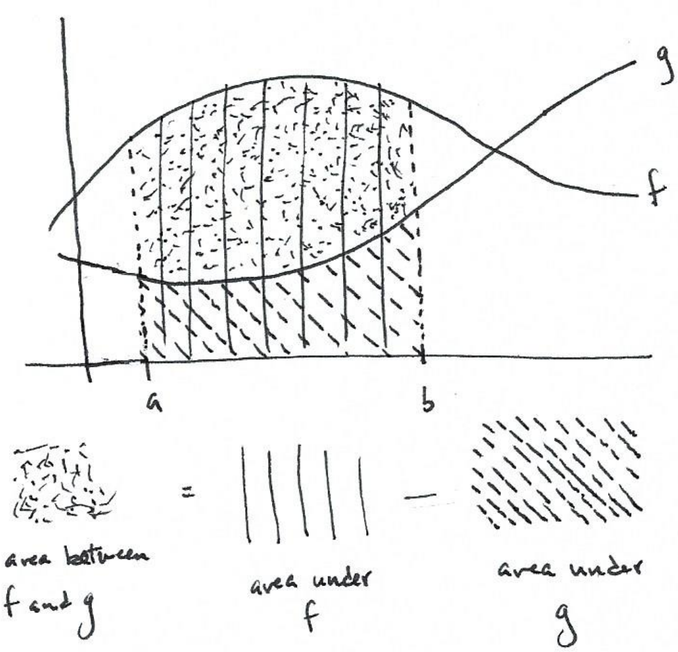
$$= \frac{\pi}{6}$$

## Application of a Region Between Two Curves

Thm. If $f$ and $g$ are continuous on $[a,b]$ and $g(x) \le f(x)$ $\forall x \in [a,b]$, then the area of the region bounded by the graphs of $f$ and $g$ and the vertical lines $\begin{cases} x = a \\ x = b \end{cases}$

is $$\int_a^b [f(x) - g(x)]\,dx$$

or $$\int_{x_1}^{x_2} \left[y(x)_{\text{top}} - y(x)_{\text{bottom}}\right] dx.$$



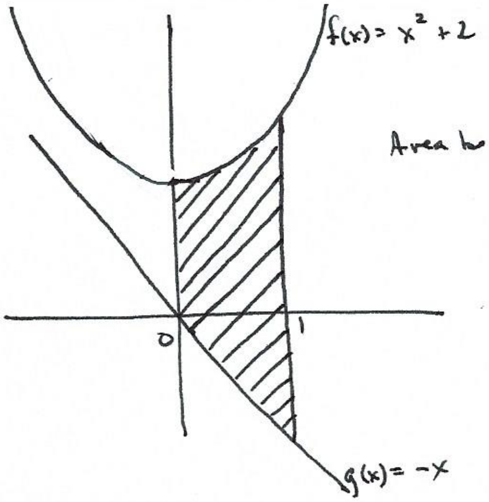

If integrating with respect to $y$, then

$$\int_c^d [f(y) - g(y)]\,dy$$

or

$$\int_{y_1}^{y_2} \left[x(y)_{\text{right}} - x(y)_{\text{left}}\right] dy$$

Ex.




Area between f and g $= \int_0^1 \left[(x^2+2) - (-x)\right] dx$

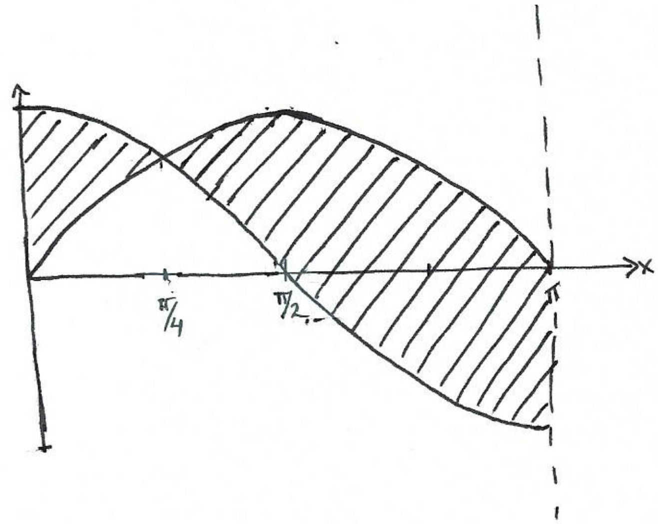
$= \left[\frac{x^3}{3} + 2x + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{3} + 2 + \frac{1}{2}$

$= \frac{17}{6}$

Ex. $f(x) = \sin(x)$, $g(x) = \cos(x)$

$x \in [0, \pi]$




Area bounded between f and g:

$$\int_0^{\pi/4} \left[\cos(x) - \sin(x)\right] dx + \int_{\pi/4}^{\pi} \left[\sin(x) - \cos(x)\right] dx$$

$$= +\sin(x)\Big|_0^{\pi/4} + \cos(x)\Big|_0^{\pi/4} - \cos(x)\Big|_{\pi/4}^{\pi} - \sin(x)\Big|_{\pi/4}^{\pi}$$

$$= +\left(\frac{\sqrt{2}}{2} - 0\right) + \left(\frac{\sqrt{2}}{2} - 1\right) - \left(-1 - \frac{\sqrt{2}}{2}\right) - \left(0 - \frac{\sqrt{2}}{2}\right)$$

$$= +\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 + 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= 2\sqrt{2}$$