Ex. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx = 2\int \left(\frac{u+3}{u}\right)(u+3)\,du = 2\int \left(1+\frac{3}{u}\right)(u+3)\,du$

let $u=\sqrt{x}-3$

$du = \dfrac{dx}{2\sqrt{x}}$

$2\sqrt{x}\,du = dx$

$2(u+3)\,du = dx$

$$= 2\int \left(u+3+3+\frac{9}{u}\right)du$$

$$= 2\left[\frac{u^2}{2}+6u+9\ln|u|\right]+C$$

$$= 2\left[\frac{(\sqrt{x}-3)^2}{2}+6(\sqrt{x}-3)+9\ln\left|\sqrt{x}-3\right|\right]+C$$

Ex Find the average value of $f(x)=\tan(x)$ on $[0,\frac{\pi}{4}]$

$$\frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4}\tan(x)\,dx = \frac{4}{\pi}\int_0^{\pi/4}\frac{\sin(x)}{\cos(x)}\,dx = -\frac{4}{\pi}\int_{x=0}^{x=\pi/4}\frac{du}{u} = -\frac{4}{\pi}\Big[\ln|u|\Big]_{x=0}^{x=\pi/4} = \frac{-4}{\pi}\ln|\cos(x)|\Big|_0^{\frac{\pi}{4}}$$

let $u=\cos(x)$

$du = -\sin(x)\,dx$

$$= \frac{-4}{\pi}\left[\ln\left|\cos\left(\tfrac{\pi}{4}\right)\right| - \ln|\cos(0)|\right]$$

$$= \frac{-4}{\pi}\left[\ln\left(\tfrac{\sqrt{2}}{2}\right) - \ln(1)\right]$$

$$= \frac{-4}{\pi}\ln\left(\tfrac{\sqrt{2}}{2}\right)$$

Ex Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x\ln(x)}$ passing through the point $(e,-1)$

$$\int \frac{dx}{x\ln(x)} = \int \frac{\frac{1}{x}dx}{\ln(x)} = \int \frac{du}{u} = \ln|u| + c = \underline{\ln|\ln(x)| + C}$$

let $u=\ln(x)$

$du = \frac{1}{x}dx$

Using the condition that $f(e)=-1$,

$$\ln|\ln(e)| + C = \ln|1| + C = C = -1$$

So $y = f(x) = \ln|\ln(x)| - 1$. $\square$

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Ex. Find the particular solution that satisfies the differential equation and the initial equations.

$$f''(x) = \frac{2}{x^2}, \quad f(-1) = 1, \quad f(1) = 4, \quad x > 0$$

$$f'(x) = \int \frac{2}{x^2}\,dx = 2\int x^{-2}\,dx = \frac{-2}{x} + C_1$$

$$f(x) = \int \left(\frac{-2}{x} + C_1\right)dx = -2\ln|x| + C_1 x + C_2$$

$$f(-1) = 1 = -2\ln|-1| + C_1(-1) + C_2 = 0 - C_1 + C_2$$

$$C_1 + 1 = C_2$$

$$f(1) = 4 = -2\ln|1| + C_1(1) + C_2 = 0 + C_1 + (C_1 + 1) = 2C_1 + 1$$

$$4 = 2C_1 + 1$$

$$\frac{3}{2} = C_1 \rightarrow \frac{3}{2} + 1 = \frac{5}{2} = C_2$$

So $f(x) = -2\ln|x| + \frac{3}{2}x + \frac{5}{2}$ □

Thm. Inverse Trig Integrals

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|x|}{a}\right) + C$$

Ex. $$\int \frac{dx}{2 + 9x^2} = \int \frac{dx}{(\sqrt{2})^2 + (3x)^2} = \int \frac{\frac{1}{3}\,du}{(\sqrt{2})^2 + u^2} = \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) + C = \frac{1}{3\sqrt{2}}\arctan\left(\frac{3x}{\sqrt{2}}\right) + C$$

let $u = 3x$
$du = 3dx$

$$\neq \frac{1}{\sqrt{2}}\arctan\left(\frac{3x}{\sqrt{2}}\right) + C$$

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Ex. $\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{dx}{2x\sqrt{x^2-\frac{9}{4}}} = \frac{1}{2}\int \frac{dx}{x\sqrt{x^2-(\frac{3}{2})^2}} = \frac{1}{2}\cdot\frac{1}{(\frac{3}{2})}\cdot \operatorname{arcsec}\left(\frac{|x|}{(\frac{3}{2})}\right)+C$

$= \frac{1}{3}\operatorname{arcsec}\left(\frac{2}{3}|x|\right)+C$

Ex $\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}(|u|)+C = \operatorname{arcsec}|e^x|+C = \operatorname{arcsec}(e^x)+C$

let $u=e^x$

$du = e^x dx$

$\frac{du}{e^x} = \frac{du}{u} = dx$

Ex. $\int \frac{x+2}{\sqrt{4-x^2}}dx = \int \frac{x}{\sqrt{4-x^2}}dx + 2\int \frac{dx}{\sqrt{4-x^2}}$, $u=4-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int \frac{du}{u^{1/2}} + 2\int \frac{dx}{\sqrt{2^2-x^2}}$

$= -\frac{1}{2}\frac{u^{1/2}}{1/2} + 2\arcsin\left(\frac{x}{2}\right)+C$

$= -(4-x^2)^{1/2} + 2\arcsin\left(\frac{x}{2}\right)+C$

Ex. $\int \frac{dx}{x^2-4x+7} = \int \frac{dx}{x^2-4x+4+3} = \int \frac{dx}{(x-2)^2+3} = \int \frac{dx}{(x-2)^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right)+C$

note: $x^2+ax+\left(\frac{a}{2}\right)^2 = \left(x+\frac{a}{2}\right)^2$

$cx^2+ax = c\left(x^2+\frac{a}{c}x\right)$

$= c\left[x^2+\frac{a}{c}x+\left(\frac{a}{2c}\right)^2\right] - c\left(\frac{a}{2c}\right)^2$

$= c\left(x+\frac{a}{2c}\right)^2 - c\left(\frac{a}{2c}\right)^2$