Integration by Substitution

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Thm. Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

This procedure is called <u>u-substitution</u> or <u>change of variables</u>.

<u>Ex</u> $\int (x+1)^{30}\,dx = \int u^{30}\,du = \frac{u^{31}}{31} + C = \frac{(x+1)^{31}}{31} + C$

let $u = x+1$

$du = dx$

<u>Ex</u> $\int (2x+1)^{30}\,dx = \int \frac{u^{30}}{2}\,du = \frac{u^{31}}{2\cdot 31} + C = \frac{(2x+1)^{31}}{62} + C$

let $u = 2x+1$

$du = 2dx$

$\frac{1}{2}du = dx$

<u>Ex.</u> $\int \frac{3x}{(1-2x^2)^2}\,dx = \int \frac{-\frac{3}{4}du}{u^2} = -\frac{3}{4}\int \frac{du}{u^2} = -\frac{3}{4}\int u^{-2}\,du = \frac{-3}{4}\,\frac{u^{-1}}{-1} + C = \frac{3}{4}(1-2x^2)^{-1} + C = \frac{3}{4(1-2x^2)} + C$

let $u = 1-2x^2$

$du = -4x\,dx$

$-\frac{1}{4}du = dx$

<u>Ex</u> $\int \sqrt{2x-1}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C = \frac{1}{2}\,\frac{2}{3}(2x-1)^{3/2} + C = \frac{1}{3}(2x-1)^{3/2} + C$

let $u = 2x-1$

$du = 2dx$

$\frac{1}{2}du = dx$

Ex $\int \sin^2(3x)\cos(3x)\,dx$ Try $u = \cos(3x)$

$du = -3\sin(3x)\,dx$

$\frac{-1}{3}du = \sin(3x)\,dx$ ⟵ in trouble here because we would need this to be $\sin^2(3x)dx$ for the substitution to work.

So let $u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$\frac{1}{3}du = \cos(3x)\,dx$

Then $\int \sin^2(3x)\cos(3x)\,dx = \frac{1}{3}\int u^2\,du = \frac{1}{3}\frac{u^3}{3} + C = \frac{\sin^3(3x)}{9} + C$

Ex. $\int \frac{5-e^x}{e^{2x}}\,dx = \int \frac{5}{e^{2x}}dx - \int \frac{e^x}{e^{2x}}dx = 5\int e^{-2x}dx - \int e^{-x}dx$

$u = -2x$, $du = -2dx$ $\quad$ $v = -x$, $dv = -dx$

$= 5\int e^u\left(\frac{-1}{2}\right)du - \int e^v(-dv)$

$= -\frac{5}{2}\int e^u\,du + \int e^v\,dv = -\frac{5}{2}e^u + e^v + C$

$= -\frac{5}{2}e^{-2x} + e^{-x} + C$

Ex. $\int \frac{2x+1}{\sqrt{x+4}}\,dx = \int \frac{2x}{\sqrt{x+4}}dx + \int \frac{dx}{\sqrt{x+4}} = 2\int \frac{x\,dx}{u^{1/2}} + \int \frac{du}{u^{1/2}} = 2\int (u-4)u^{-1/2}du + \int u^{-1/2}du$

let $u = x+4$
$du = dx$
$x = u-4$

$= 2\int u\cdot u^{-1/2}du - 2\int 4u^{-1/2}du + \int u^{-1/2}du$

$= 2\int u^{1/2}du - 8\int u^{-1/2}du + \int u^{-1/2}du$

$= 2\int u^{1/2}du - 7\int u^{-1/2}du$

$= 2\frac{u^{3/2}}{3/2} - 7\frac{u^{1/2}}{1/2} + C$

$= \frac{4}{3}(x+4)^{3/2} - 14(x+4)^{1/2} + C$

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Ex Find an equation for the function f that has derivative $f'(x) = x^2 e^{-x^3/5}$ and whose graph passes through $(0, 3/2)$.

$$f(x) = \int x^2 e^{-x^3/5}\,dx = -\frac{5}{3}\int e^u\,du = \frac{-5}{3}e^u + C = -\frac{5}{3}e^{-x^3/5} + C$$

let $u = \frac{-x^3}{5}$

$du = \frac{-3}{5}x^2\,dx$

$$f(0) = \frac{3}{2} = -\frac{5}{3}e^{-(0)^3/5} + C = \frac{-5}{3} + C \longmapsto C = \frac{3}{2} + \frac{5}{3} = \frac{9+10}{6} = \frac{19}{6}$$

$$\Longrightarrow f(x) = -\frac{5}{3}e^{-x^3/5} + \frac{19}{6}$$

Thm If the function $u = g(x)$ has continuous derivative on $[a,b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Ex $$\int_1^4 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_3^{3/4} e^u\,du = \frac{1}{3}\int_{3/4}^{3} e^u\,du = \frac{1}{3}e^u\Big|_{3/4}^{3} = \frac{1}{3}\left(e^3 - e^{3/4}\right)$$

let $u = \frac{3}{x}$ $\quad$ $x=1 \to u=3$

$du = \frac{-3}{x^2}\,dx$ $\quad$ $x=4 \to u=\frac{3}{4}$

$$= \frac{1}{3}e^{\frac{3}{x}}\Big|_{3/4}^{1} = \frac{1}{3}\left(e^3 - e^{3/4}\right)$$