Ex. Find the values of c guaranteed by the Mean Value Theorem for Integrals

a) $f(x) = \cos(x)$ on $\left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos(x)\,dx = f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right]$$

$$\sin(x)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \cos(c)\cdot\frac{2\pi}{3}$$

$$\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) = \cos(c)\cdot\frac{2\pi}{3}$$

$$\sqrt{3} = \cos(c)\cdot\frac{2\pi}{3}$$

$$\frac{3\sqrt{3}}{2\pi} = \cos(c)$$

$$c = \arccos\left(\frac{3\sqrt{3}}{2\pi}\right)$$

Def. If f is integrable on $[a,b]$, then the average value of f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex Find the average of $f(x) = 9 - x^2$ on $[-3,3]$ and all values of x in $[-3,3]$ for which the function equals its average value.

$$\frac{1}{3-(-3)}\int_{-3}^{3}(9-x^2)\,dx = \frac{1}{6}\left(9x - \frac{1}{3}x^3\right)\Big|_{-3}^{3} = \frac{1}{6}\left[(27-(-27)) - \frac{1}{3}(27-(-27))\right]$$

$$= \frac{1}{6}\left[54 - \frac{1}{3}\cdot 54\right] = \frac{1}{6}\cdot\frac{2}{3}\cdot 54 = 6$$

$f(x) = 6$, $x \in [-3,3]$

$$9 - x^2 = 6$$

$$x^2 = 3$$

$x = \pm\sqrt{3}$ ← since both values in the interval, they are both solutions

Math 2413
Dr. Liu
13 Nov 2018
pg 80

<u>Ex.</u> At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ in meters per second can be modeled by

$$s(x) = \begin{cases} -4x+341, & 0 \le x \le 11.5 \\ 295, & 11.5 \le x \le 22 \end{cases}$$ where x is the altitude in kilometers.

what is the average speed of sound over the interval $[0,22]$?

$$\frac{1}{22-0}\int_0^{22} s(x)\,dx = \frac{1}{22}\left[\int_0^{11.5}(-4x+341)\,dx + \int_{11.5}^{22}(295)\,dx\right]$$

$$= \frac{1}{22}\left[\left(-4\cdot\frac{x^2}{2}+341x\right)\Big|_0^{11.5} + (295x)\Big|_{11.5}^{22}\right]$$

$$= \frac{1}{22}\left[-2(11.5)^2 + 341(11.5) - (0+0) + 295(22-11.5)\right]$$

$$\approx 307 \text{ m/s}$$

<u>Thm.</u> <u>Second Fundamental Theorem of Calculus</u>

If f is continuous on an open interval I containing a, then for every $x \in I$,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

In general, we have

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

<u>Ex.</u> $\frac{d}{dx}\left[\int 1\,dx\right] = \frac{d}{dx}\left[x+c\right] = 1$

<u>Ex</u> $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)\cdot x' - f(a)\cdot a'$

$$= f(x)(1) - f(a)\cdot(0)$$

$$= f(x)$$

<u>Ex.</u>

$$\frac{d}{dx}\left[\int_{-x}^{3x+2}\sqrt{t^2+1}\,dt\right] = f(3x+2)\frac{d}{dx}(3x+2) - f(-x)\frac{d}{dx}(-x)$$

$f(x) = (t^2+1)^{1/2}$

$$= \left[(3x+2)^2+1\right]^{1/2}(3) - \left[(-x)^2+1\right]^{1/2}(-1)$$

$$= 3\left[(3x+2)^2+1\right]^{1/2} + \left[x^2+1\right]^{1/2}$$

Math 2413
Dr. Liu
13 Nov 2018
pg 81

Ex. $F(x) = \int_{\pi/2}^{x^3} \cos(t)\,dt$

a) Integrate and find $F'(x)$

b) Demonstrate Second Fundamental Theorem of Calculus by finding derivative of $F(x)$.

a) $\frac{d}{dx}\left[\int_{\pi/2}^{x^3} \cos(t)\,dt\right] = \frac{d}{dx}\left[\sin(t)\Big|_{\pi/2}^{x^3}\right] = \frac{d}{dx}\left[\sin(x^3) - \sin\left(\frac{\pi}{2}\right)\right] = \frac{d}{dx}\left[\sin(x^3) - 1\right]$

$= 3x^2\cos(x^3)$

b) $\frac{d}{dx}\left[\int_{\pi/2}^{x^3} \cos(t)\,dt\right] = \cos(x^3)\cdot\frac{d}{dx}(x^3) - \cos\left(\frac{\pi}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{2}\right) = \cos(x^3)\cdot(3x^2) - 0\cdot 0$

$= 3x^2\cos(x^3)$

Thm: The definite integral of the rate of change of quantity $F'(x)$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

gives the total or <u>net change</u> in that quantity over the interval $[a,b]$.

Ex. A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute, where $0 \le t \le 60$ is the time in minutes. Find the amount of the chemical that flows into the tank during first 20 minutes.

Let $c(t)$ be the amount of chemical in tank at time t.

Then $c'(t)$ is the chemical flow rate at time t.

$$\int_0^{20} c'(t)\,dt = \int_0^{20} (180+3t)\,dt = \underbrace{\left(180t + \frac{3}{2}t^2\right)}_{c(t)}\Big|_0^{20} = 180(20) + \frac{3}{2}(20)^2 = 3600 + \frac{3}{2}\cdot 400 = 4200$$