Def: let $f$ be differentiable at $c$. The linear approximation of $f$ at $c$ is

$$y = f(c) + f'(c)(x-c)$$

or tangent line approximation.

Ex. Find the tangent line approximation of $f(x) = \sqrt{x}$ at $x=16$. Use this to approximate $\sqrt{16.5}$.

$$f(x) = x^{1/2}$$
$$f'(x) = \frac{1}{2}x^{-1/2}$$
$$f'(16) = \frac{1}{2}4^{-1/2} = \frac{1}{8}$$
$$f(16) = 4$$

$$y = f(16) + f'(16)(x-16) = 4 + \frac{1}{8}(x-16) = \frac{1}{8}(x) + 2$$

$$\sqrt{16} \approx \frac{x}{8} + 2 \text{ near } x = 16$$

$$\frac{1.5}{8} + 2 = \frac{16}{8} + \frac{1}{8} + 2 = 4 + \frac{1}{8} = 4.0625$$

Def: let $y = f(x) + f'(x)(x-c)$, $(x-c)$ is called the change in $x$ and is denoted $\Delta x$.

when $\Delta x$ is small, $\Delta y$ can be approximated as

$$\Delta y = f(c+\Delta x) - f(c) \approx f'(c)\Delta x$$

Def: let $y = f(x)$ be differentiable on an open interval containing $x$. The differential of $x$ is any nonzero real number. The differential of $y$ is

$$dy = f'(x)dx$$

Ex. Let $y = \sqrt{x}$. Find $dy$ when $x=16$ and $dx = 0.5$. Compare with $\Delta y$ for $x=16$ and $dx = 0.5$.

$$dy = f'(x)dx = \frac{1}{2\sqrt{x}} \cdot \frac{1}{2} = \frac{1}{4\sqrt{16}} = \frac{1}{16}$$

$$\Delta y = \sqrt{16.5} - \sqrt{16} \approx 1.0625 - 4 = 0.0625$$

Note: in many applications, $dy$ is used to approximate $\Delta y$, such as in error propagation for physical measuring devices.

$$\frac{f(x + \Delta x) - f(x)}{\Delta y} = \frac{\text{error propagated}}{\text{exact value}}$$
Ex. The measured radius of a ball bearing is 0.7 inch, correct to within 0.01 inch. Estimate the propagated error in the volume \( V \) of the ball bearing. Is the propagated error small or large?

\[
V = \frac{4}{3} \pi r^3, \quad r = 0.7, \quad -0.01 \leq \Delta r \leq 0.01
\]

\[
\Delta V = V(0.7 + \Delta r) - V(0.7) \approx dV = 4\pi r^2 dr = 4\pi (0.7)^2 (0.01) \approx 0.06158
\]

The volume has a propagated error of about 0.06158 cubic inch.

Relative error = \[
\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3} \pi r^3} = \frac{3dr}{r} = \frac{3}{0.7} (0.01) \approx 0.0429.
\]

The percent error in volume measurement is approximately 4.29%.

Ex. Find differential of \( y = \sin(3x) \)

\[
dy = \cos(3x) (3x)' dx
\]

\[
= 3\cos(3x) dx
\]