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Note Consider $\lim_{x\to a}[f(x)]^{g(x)}$. Several indeterminate forms arise.

1) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0 \longmapsto 0^0$ type

2) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0 \longmapsto \infty^0$ type

3) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty \longmapsto 1^\infty$ type

In these cases, we either i) let $y = [f(x)]^{g(x)}$, then $\ln(y) = g(x)\ln[f(x)]$

ii) $[f(x)]^{g(x)} = e^{g(x)\ln[f(x)]}$

Both methods lead to indeterminate product $g(x)\ln[f(x)]$ of $0\cdot\infty$ type.

Ex Calculate the limit.

a) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, s

b/c $\ln(x)$ continuous

$$\text{so } \ln(y) = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right] = \lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x}\right)^x\right] = \underbrace{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)}^{\infty\cdot 0}$$

$$= \underbrace{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}}_{\frac{0}{0}} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

Since $\ln(y) = 1$, $y = e$

$$\Longrightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

Ex (cont)

b) $\lim_{x \to 0^+} (\sin(x))^x = y$

because ln is continuous

$$\ln(y) = \ln\left[\lim_{x\to 0^+}(\sin(x))^x\right] = \lim_{x\to 0^+} \ln\left[(\sin(x))^x\right] = \lim_{x\to 0^+} \underbrace{x \ln[\sin(x)]}_{0\cdot(-\infty)} = \lim_{x \to 0^+} \frac{\ln[\sin(x)]}{\frac{1}{x}}$$

$$\stackrel{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin(x)}\cdot\cos(x)}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \underbrace{\frac{-x^2}{\tan(x)}}_{\left(\frac{0}{0}\right)} \stackrel{LH}{=} \lim_{x\to 0^+} \frac{-2x}{\sec^2(x)} = \lim_{x\to 0^+} -2x\cos^2(x)$$

$$= 0$$

Since $\ln(y) = 0$, $y = \lim_{x\to 0^+} [\sin(x)]^x = 1$

A summary of Curve Sketching

1) Determine domain and range of function
2) Determine intercepts, asymptotes, symmetries of graph
3) Locate x-values for which $f'(x)$ and $f''(x)$ are either zero or do not exist

Ex. Sketch the graph.

a) $\boxed{f(x) = \frac{2(x^2-9)}{x^2-4}}$

① Domain: $\underbrace{x \neq \pm 2}_{\text{vertical asymptotes}}$

② Range: $(\mathbb{R} \setminus \{2\}) = (-\infty, 2) \cup (2, \infty)$

③ x-intercept: let $\frac{2(x^2-9)}{x^2-4} = 0$

$x^2 - 9 = 0$

$x = \pm 3$

④ y-intercept: let $x = 0$

$\frac{2(-9)}{-4} = \frac{9}{2}$

⑤ Horiz asymptotes:

$$\lim_{x\to\infty} \frac{2(x^2-9)}{x^2-4} \stackrel{LH}{=} \lim_{x\to\infty} \frac{4x}{2x} = 2$$

$$\lim_{x\to -\infty} \frac{2(x^2-9)}{x^2-4} \stackrel{LH}{=} \lim_{x\to -\infty} \frac{4x}{2x} = 2$$

⑥ $f'(x) = \frac{4x(x^2-4) - 2(x^2-9)(2x)}{(x^2-4)^2} = \frac{20x}{(x^2-4)^2}$

$f'(x) = 0 \mapsto x = 0 \leftarrow$ crit point

$f'(x)$ DNE $\mapsto x = \pm 2$, but $f(\pm 2)$ undefined

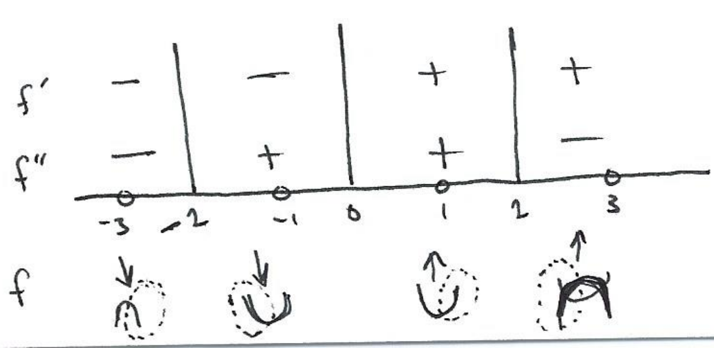
⑦ $f''(x) = \frac{20(x^2-4)^2 - 20x[2(x^2-4)2x]}{(x^2-4)^4}$

$= \frac{-20(3x^2+4)}{(x^2-4)^4}$

$f''(x) = 0 \mapsto 3x^2 + 4 = 0 \to$ NO SUCH x

$f''(x)$ DNE $\mapsto x = \pm 2$, but $f(\pm 2)$ undefined

$\Longrightarrow$ no inflection points



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