Indeterminate Forms and L'Hôpital's Rule

Math 2413
Dr. Lin
16 Oct 2018
pg 52

Def $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ is in <u>indeterminate form</u> if $\begin{cases} \displaystyle\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x), & \left(\frac{0}{0}\right) \\ \displaystyle\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x), & \left(\frac{\infty}{\infty}\right) \end{cases}$

thm In this case, if f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a),

then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Ex Evaluate the limits

a) $\displaystyle\lim_{x\to 0} \underbrace{\frac{e^{2x}-1}{x}}_{\frac{0}{0}} \overset{LH}{=} \lim_{x\to 0} \frac{2e^{2x}-0}{1} = 2$

b) $\displaystyle\lim_{x\to \infty} \underbrace{\frac{\ln(x)}{x}}_{\frac{\infty}{\infty}} \overset{LH}{=} \lim_{x\to \infty} \frac{\frac{1}{x}}{1} = 0$

c) $\displaystyle\lim_{x\to -\infty} \frac{x^2}{\underbrace{e^{-x}}_{\frac{\infty}{\infty}}} \overset{LH}{=} \lim_{x\to -\infty} \frac{2x}{\underbrace{-e^{-x}}_{\frac{-\infty}{-\infty}}}$

$\displaystyle\overset{LH}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}} = 0$

Def $\displaystyle\lim_{x\to a} f(x)g(x)$ is indeterminate if $\displaystyle\lim_{x\to a} f(x) = 0$ and $\displaystyle\lim_{x\to a} g(x) = \pm\infty$, $(0\cdot\infty)$

In this case, we convert to $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form by rewriting

fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$

Ex. Evaluate the limits

a) $\displaystyle\lim_{x\to \infty} \underbrace{e^{-x}\sqrt{x}}_{(0\cdot\infty)} = \lim_{x\to \infty} \underbrace{\frac{\sqrt{x}}{e^x}}_{\left(\frac{\infty}{\infty}\right)} \overset{LH}{=} \lim_{x\to \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to \infty} \frac{1}{2\sqrt{x}\,e^x} = 0$

b) $\displaystyle\lim_{x\to 0^+} \underbrace{x\ln(x)}_{(0\cdot(-\infty))} = \lim_{x\to 0^+} \underbrace{\frac{\ln(x)}{\frac{1}{x}}}_{\left(\frac{\infty}{\infty}\right)} \overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-1}{\frac{1}{x}} = \lim_{x\to 0^+} -x = 0$

Def $\lim_{x\to a} f(x) - g(x)$ is indeterminate of type $(\infty - \infty)$ if $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$

In this case, convert difference to quotient by finding common denominator, rationalization, or factoring. This gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form

Ex. Evaluate the limit

a) $\lim_{x\to \frac{\pi}{2}^-} \sec(x) - \tan(x) = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin(x)}{\cos(x)} \overset{LH}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos(x)}{-\sin(x)} = \frac{0}{1} = 0$

b) $\lim_{x\to 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln(x)}{x\ln(x) - \ln(x)} \overset{LH}{=} \lim_{x\to 1^+} \frac{1 - \frac{1}{x}}{\ln(x) + x\cdot\frac{1}{x} - \frac{1}{x}} = \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\ln(x) + 1 - \frac{1}{x}}$

$\overset{LH}{=} \lim_{x\to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1+1} = \frac{1}{2}$