**Composition of Functions**

- **Introductory Example**
  \[ f(x) = 2x + 3 \quad g(x) = x^2 \]
  Recall \[ f(2) = 2(2) + 3 = 7 \]
  \[ f(7) = 2(7) + 3 = 17 \]

  If I want to evaluate the function \( f(x) \) at \( g(x) \),
  \[ f(g(x)) = 2(g(x)) + 3 \]
  \[ = 2x^2 + 3 \]

  We denote \( f(g(x)) \) by \((f \circ g)(x)\), "\( f \) of \( g \) of \( x \)". This is the composition of functions.

- **Definition**
  \[(f \circ g)(x) = f(g(x))\]

- **Example**
  \[ f(x) = 3x + 5 \quad g(x) = 2x - 7 \]

  1. \((f \circ g)(x) = 3(g(x)) + 5 = 3(2x - 7) + 5 = 6x - 21 + 5 = 6x - 16\)

  2. \((g \circ f)(x) = 2(f(x)) - 7 = 2(3x + 5) - 7 = 6x + 10 - 7 = 6x + 3\)

  Notice order matters! \((f \circ g)(x) \neq (g \circ f)(x)\) in general.
The composition of functions is NOT commutative!

3) Evaluate \((f \circ g)(x)\) at \(x = 3\)
\[
(f \circ g)(3) = 6(3) - 16 = 18 - 16 = 2
\]

Evaluate \((g \circ f)(x)\) at \(x = 3\)
\[
(g \circ f)(3) = 6(3) + 3 = 18 + 3 = 21
\]

Hawkes Practice:

1) Find \((f \circ g)(1)\) for the following functions
\[
f(-10) = 12, \quad g(1) = -10.
\]
Recall \((f \circ g)(1) = f(g(1))\)
\[
\text{plug in } \quad \text{Since } g(1) = -10
\]
\[
= f(-10) \quad \text{Because } f(-10) = 12.
\]
\[
= 12
\]

2) Find \((f \circ g)(1)\) for the following functions. Round your answers to 2 decimal places if necessary.
\[
\begin{align*}
 f(x) &= 4 + \sqrt{x} \quad g(x) = \sqrt{x^2 + 15}.
\end{align*}
\]
\[
(f \circ g)(1) = f(g(1))
\]
\[
g(1) = \sqrt{1^2 + 15} = \sqrt{16} = 4
\]
\[ f(g(x)) = f(4) \]
\[ = 4 + \sqrt{4} \]
\[ = 4 + 2 \]
\[ = 6. \]

\[ (f \circ g)(1) = 6 \]

(3) Given \( f(x) \), find \( g(x) \) and \( h(x) \) such that \( f(x) = g(h(x)) \) and neither \( g(x) \) nor \( h(x) \) is solely \( x \).

\[ f(x) = \sqrt{-x^2 - 2} - 5. \]

Start by identifying a parent function.

\[ g(x) = \sqrt{x} - 5. \]

Insert a function to replace \( x \) such that you have \( f(x) \).

\[ h(x) = -x^2 - 2. \]

Then
\[ g(h(x)) = \sqrt{h(x)} - 5 \]
\[ = \sqrt{-x^2 - 2} - 5. \]

\[ \therefore f(x) = g(h(x)) \]

(4) Given \( f(x) \), find \( g(x) \) and \( h(x) \) such that \( f(x) = g(h(x)) \).

\[ f(x) = \frac{\sqrt{3x-1}}{2} \]

\[ g(x) = \frac{\sqrt{x}}{2}, \quad h(x) = 3x-1 \]

\[ \therefore g(h(x)) = \frac{\sqrt{h(x)}}{2} = \frac{\sqrt{3x-1}}{2} = f(x) \]
Consider the following functions.

\[ f(x) = -5x, \quad g(x) = \frac{3}{\sqrt{x}} \]

0. Find the formula for \((f+g)(x)\) and simplify your answer.

\[
(f+g)(x) = f(x) + g(x) = -5x + \frac{3}{\sqrt{x}}.
\]

1. What's the domain of \((f+g)(x)\)?

There is no limitations for this function, because there is no square root and no fractions.

So the domain is all real numbers. \(\mathbb{R}\).

2. Find the formula for \((\frac{f}{g})(x)\) and simplify.

\[
(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{-5x}{\frac{3}{\sqrt{x}}} = \frac{-5x}{\sqrt[3]{x}}.
\]

3. What's the domain of \((\frac{f}{g})(x)\)?

The denominator cannot be zero.

Set \(\sqrt[3]{x} = 0\), solve for \(x\), \(x = 0\).

So exclude zero from your domain.

There is no radical with even index.

Therefore the domain is all real numbers except 0.

Interval notation \((-\infty, 0) \cup (0, \infty)\).
(6) Does \( x = -9 \) solve the following polynomial equation?

\[-70x^2 + 8x = 8x^3 + 111.\]

Simply plug in \( x = -9 \) and see if left side equals right side:

\[-70 \cdot (-9)^2 + 8(-9) = -70 \cdot 81 - 72 = -5670 - 72 = -5742.\]

\[8(-9)^3 + 111 = 8(-729) + 111 = -5832 + 111 = -5721.\]

\[-5742 \neq -5721.\]

\[\therefore -9 \text{ is not a solution.}\]

---

To solve a polynomial for its roots:

1. Set the polynomial to zero.

2. Factor the polynomial.

The property of zeros:

If \( a \cdot b = 0 \), then either \( a = 0 \) or \( b = 0 \).

E.g. \( x^2 + 6x + 5 \) find the solution to this polynomial:

\[x^2 + 6x + 5 = 0.\]

\[(x+5)(x+1) = 0.\]

Either \( x + 5 = 0 \) or \( x + 1 = 0 \)

\( \therefore x = -5 \) or \( x = -1 \)
If you are asked if something is a solution or not, just plug the number into the equation and see if you get a true statement.

Is $x=2$ a solution to $x^2 + 6x + 5 = 0$?

Substitute $x=2$ into $x^2 + 6x + 5$.

$$2^2 + 6(2) + 5 = 4 + 12 + 5 = 21 \neq 0$$

Therefore $x=2$ is NOT a solution.